The Crank-Nicolson Method for Solving the Heat Equation

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I. Abstract

The Crank-Nicolson (CN) method of numerical computation has consistently seen steady use for almost a century due to its versatility and consistency, especially in the matter of solving the heat equation. This paper overviews the essential function and operation of the method, and presents some examples written in MATLAB.

II. Background

The CN method was originally proposed in 1946 as a new way to numerically evaluate partial differential equations (PDEs), with respect to problems of heat flow and generation. In the method's original treatment, the PDE example used is as follows [1]:

$$\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} - q\frac{\partial w}{\partial t} \qquad (1)$$

$$\frac{\partial w}{\partial t} = -kwe^{-A/\theta} \qquad (2)$$

The proposed CN method would utilize finite difference ratios replacing both time and space derivatives. Replacing both derivatives of equation (1) at point $(m\delta x, \left(n+\frac{1}{2}\right)\delta t)$, gives the form [1]:

$$\theta_m(n+1) - \theta_m(n) = \frac{\delta t}{2(\delta x)^2}[\theta_{m-1}(n+1) + \theta_{m+1}(n+1) + \theta_{m-1}(n) + \theta_{m+1} + \theta_{m+1}(n) - 2\{\theta_m(n+1) + \theta_m(n)\}] - q[w_m(n+1) - w_m(n)]$$

(3)

In this way the solution can now be numerically solved in discrete time steps. The most common way to visualize this is with a grid overlapping an x-y graph, with time on the y-axis and space on the x-axis [2].

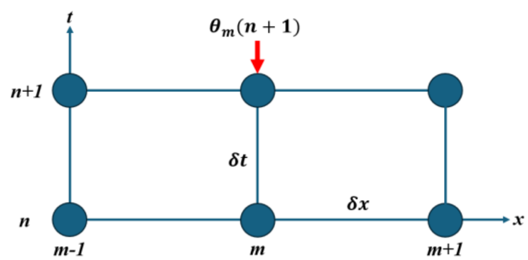

Figure 1: Visualization of CN method.

Utilizing equation (3), and all six points of the grid in Figure 1, the temperature at each time and space step can be numerically solved.

The CN method differed from others in that it replaced both time and space derivates with finite difference equations, and the steps in time do not overlap [1]. In previously proposed methods, overlapping time steps would give errors manifesting as oscillations in the solution. This would limit the size of time steps used as the larger the time step, the more oscillation is seen. With the CN method having time steps that do not overlap, much larger intervals can be used without oscillatory error. The so-called explicit method (replacing the time derivative) suffers from needing a very fine time mesh, which is for many situations not suitable for analysis and may take considerable time [1] [2]. This is manifested in needing a very small $\delta t/(\delta x)^2$. Meanwhile, the main issue with the implicit method (replacing the space derivative) was in the high amount of compute necessary and inaccuracies under certain conditions for the time when computers were few and barebones [1]. The CN method was faster than the implicit method while allowing for a higher $\delta t/(\delta x)^2$ compared to the explicit method.

III. APPLICATIONS AND EXAMPLES

An interesting point on the versatility of the CN method is in the various ways the system can be set up. For example, the boundaries may exhibit Dirichlet conditions, where they are held to a specific value, or they may be allowed to change freely after the initial condition is set. This allows for a variety of practical systems to be simulated.

For example, the first example of the CN method by [1] had the below boundary conditions for a 1D system whose length was 1.

$$\begin{cases} \theta = \text{constant at } t = 0 \text{ for } 0 \le x \le 1, \\ w = \text{constant at } t = 0 \text{ for } 0 \le x \le 1, \\ \dfrac{\partial \theta}{\partial x} = H_1(\theta) \text{ for } t \ge 0, x = 0, \\ \dfrac{\partial \theta}{\partial x} = 0 \text{ for } t \ge 0, x = 1. \end{cases}$$

In these boundary conditions, $H_1(\theta)$ is the heat transfer function, which dictates how much some external θ_0 permeates through the material. This means that between the 0^{th} node and the 1^{st} node in the spatial axis, their difference in each calculation is dictated by the heat transfer function specified. Meanwhile, at the n^{th} and $n\text{-}1^{st}$ nodes, the differential is 0, meaning that it is entirely dependent on the previous time iteration and whatever internal chemical reaction is happening thanks to the w term. Evidently, the method is very malleable, as this example basically says that the central nodes are computed as normal, however the edges are computed with modifications to some of the terms.

Figure 2 shows this situation for the halfway-to and the center of the structure, where $x = ½$ is halfway-to and $x = 1$ is the center. The solution with less x steps suffers from less numerical stability as the method tries to overcompensate due to the large gaps between mesh points. [1] states that for decent numerical stability, the $\delta t/(\delta x)^2$ ratio should not surpass 4, and this includes the diffusivity constant D that gets multiplied by the ratio.

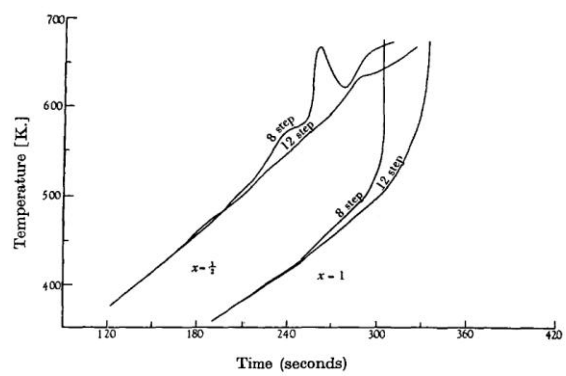

Figure 2: Temperature-time curves for x = ½ and x = 1, where the x grid is calculated with 8 and 12 steps. [1]

To show the versatility, the next few sections are several examples of the CN method in action via simulation. It is important to note that the example computed by [1] involved a boundary condition being a source of heat generation, hence the increasing of temperature across the structure in Figure 2. However, the examples produced in the next sections generally involve strict heat dissipation and thus cooling of the surface over time, with heat inputted only for the initial condition.

1. *Example 1: A single-moment point of thermal flux*

In this example, the initial conditions simply involve a point of heat being applied somewhere onto the structure. The boundary conditions are simply a diffusivity of 1. The mesh is also set up so that the $\delta t/(\delta x)^2$ ratio equals 1. Therefore, all of the conditions seem as below:

$$\begin{cases} \theta = 1 \text{ at } t = 0 \text{ and } x = 0 \\ \theta = 0 \text{ otherwise} \\ \dfrac{\partial \theta}{\partial x} = 1 \text{ for } t \geq 0, x = 0 \text{ and } 0.9 \end{cases}$$

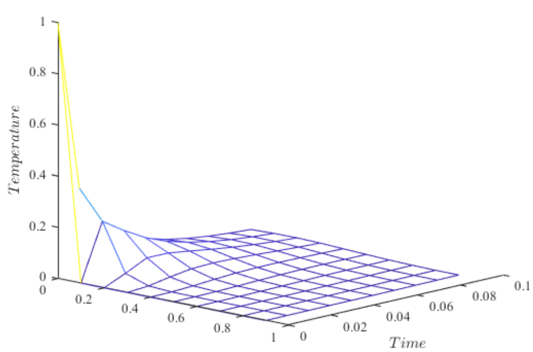

Figure 3: Spatial-Transient mesh plot of Example 1

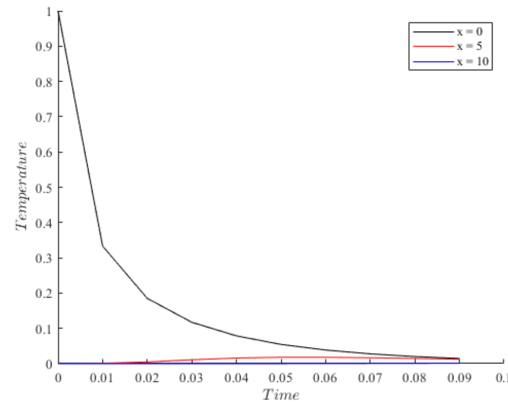

Figure 4: Temperature-Transient plot of Example 1 across three spatial slices (Note that x in the legend refers to which x slice number, rather than where the x slice is. This is true for all subsequent similar plots)

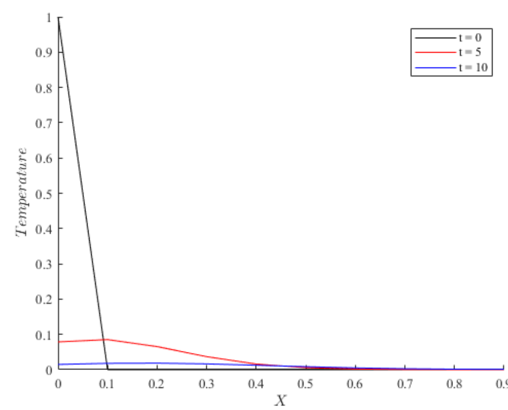

Figure 5: Temperature-Spatial plot of Example 1 across three time slices (Note that t in the legend refers to which t slice number, rather than when the t slice is. This is true for all subsequent similar plots)

As seen, with time the entire structure reaches an equilibrium of about a hundredth of the original injected temperature. The final result is about a hundredth since it spreads quadratically, thanks to the square on the δx term.

2. *Example 2: Reducing Diffusivity in the single point of thermal flux*

This situation is identical to Example 1: A single-moment point of thermal flux except that D is reduced to 0.5, producing the below conditions:

$$\begin{cases} \theta = 1 \text{ at } t = 0 \text{ and } x = 0 \\ \theta = 0 \text{ otherwise} \\ \dfrac{\partial \theta}{\partial x} = 0.5 \text{ for } t \geq 0, x = 0 \text{ and } 0.9 \end{cases}$$

These conditions affect the system by retaining the heat for longer near the right surface, as is most directly comparable using Figure 5 and Figure 8, where it can be seen that as $t = 5$, $x \approx 0.1$ and 0.2, respectively.

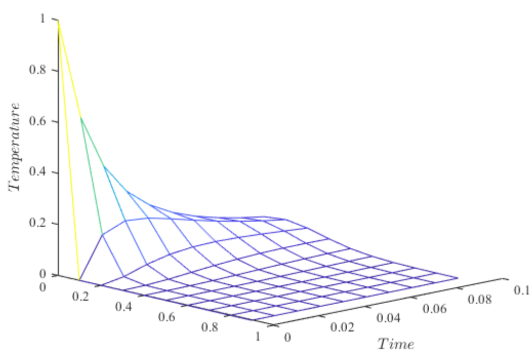

Figure 6: Spatial-Transient mesh plot of Example 2

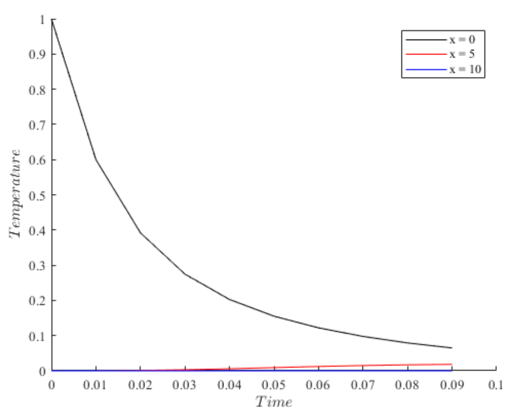

Figure 7: Temperature-Transient plot of Example 2 across three spatial slices

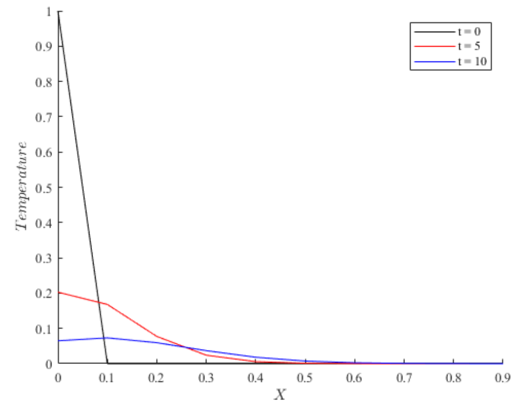

Figure 8: Temperature-Spatial plot of Example 2 across three time slices

3. *Example 3: Actively Applying Some Temperature at One Edge (Dirichlet Boundary Condition)*

In this example, let the initial condition be the first quarter of a cosine wave, D equals 1, and the right boundary condition is kept at 0, as so:

$$\begin{cases} \theta = \cos\left(\dfrac{x\pi}{20}\right) \text{ at } t = 0 \text{ for } 0 \leq x \leq 0.9 \\ \dfrac{\partial \theta}{\partial x} = 1 \text{ for } t \geq 0 \text{ at } x = 0 \\ \dfrac{\partial \theta}{\partial x} = \dfrac{\partial \theta}{\partial t} = 0 \text{ for } t \geq 0 \text{ at } x = 0.9 \end{cases}$$

This creates the effect of having the right side constantly at θ equals 0, while allowing the left edge to modulate according to how heat dissipates and spreads away from it.

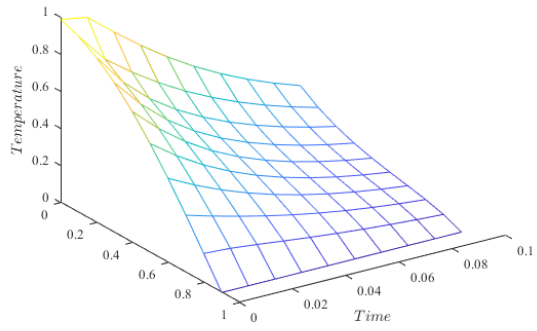

Figure 9: Spatial-Transient mesh plot of Example 3

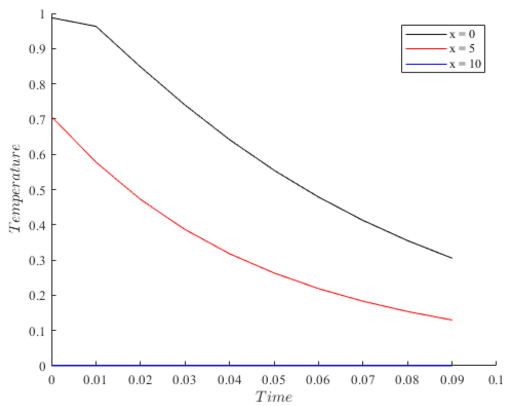

Figure 10: Temperature-Transient plot of Example 3 across three spatial slices

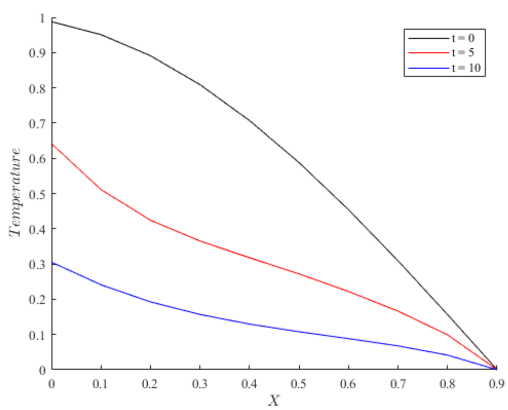

Figure 11: Temperature-Spatial plot of Example 3 across three time slices

As is evident, especially from Figure 11, with time, this situation causes the temperature to drop with time across the structure towards the temperature it is being held at, however with a gradient where the initially heated edge is always higher.

IV. Conclusions

In total, the CN method can see various uses and scenarios, without compromising in numerical stability. The situation which causes instability comes down to improperly meshing the problem in the time and space domains, which is only logical as a lack of detail will naturally produce errors in any method.

References

[1] J. Crank and P. Nicolson, "A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type," *Cambridge Philosophical Society, vol. 43, no. 1,* pp. 50-57, 1947.

[2] E. Kreyszig, Advanced Engineering Mathematics, John Wiley & Sons, 2019.